International Journal of Theoretical Physics, Vol. 38, No. 9, 1999

Evolution Equations for Lévy Stable Processes

V. V. Uchaikin[1]

Received February 12, 1999

Evolution equations of one-dimensional Lévy stable processes are derived in terms of fractional integrodifferentiation operators, generalizing the Einstein diffusion equation for the Wiener process, and applicable to the description of anomalous diffusion.

1. INTRODUCTION

A homogeneous random process $X(t; \alpha, \beta)$ with independent increments is called a Lévy stable process if its increments are distributed according to a Lévy stable law (Feller, 1971). We consider the strictly stable processes obeying the condition

$$X(t; \alpha, \beta) \overset{d}{=} t^{1/\alpha} Y(\alpha, \beta), \qquad t > 0 \tag{1}$$

where $Y(\alpha, \beta)$ is the standardized strictly stable variable with characteristic exponent $\alpha \in (0, 2]$ and asymmetry β (Zolotarev, 1986). The symbol $\overset{d}{=}$ in Eq. (1) means that both left side and right side have the same distribution.

In terms of distribution densities the relation (1) is expressed as follows:

$$p(x, t; \alpha, \beta) = t^{-1/\alpha} g(xt^{-1/\alpha}; \alpha, \beta) \tag{2}$$

where p and g stand for the probability densities of random variables X and Y, respectively.

If $\alpha = 2$, then the random variable $Y(2, 0)$ is distributed according to the Gaussian law

[1] Ulyanovsk State University, Ulyanovsk 432700, Russia; e-mail: uchaikin@sv.uven.ru.

0020-7748/99/0900-2377$16.00/0

$$g(x;\, 2,\, 0) = \frac{1}{2\sqrt{\pi}}\, e^{-x^2/4}$$

and the process (1) is merely the Wiener process, the distribution density of which

$$p(x,\, t;\, 2,\, 0) = \frac{1}{2\sqrt{\pi t}}\, e^{-x^2/4t} \tag{3}$$

satisfies the evolution equation

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} \tag{4}$$

under the initial condition

$$p(x,\, 0;\, 2,\, 0) = \delta(x)$$

Equation (4) is the well-known diffusion equation derived by Einstein for description of Brownian motion.

Insofar as the Wiener process is a special kind of Lévy stable process, it is natural to seek a generalization of the evolution equation (4) to the whole family of Lévy stable processes. This is the aim of this paper.

2. EVOLUTION EQUATIONS FOR CHARACTERISTIC FUNCTIONS

Because stable densities, except for the Gauss density ($\alpha = 2$, $\beta = 0$), the Cauchy density ($\alpha = 1$, $\beta = 0$), and the Smirnov (or Lévy) density ($\alpha = 1/2$, $\beta = 1$), are not expressed in terms of elementary functions, we have to begin with their characteristic functions

$$\varphi(k,\, t;\, \alpha,\, \beta) = \int_{-\infty}^{\infty} e^{ikx} p(x,\, t;\, \alpha,\, \beta)\, dx$$

expressible in a simple form. There exist several representations of stable laws; we consider here the two most popular, forms *A* and *C* (Zolotarev, 1986). The corresponding characteristic functions have the form

$$\varphi_A(k,\, t;\, \alpha,\, \beta) = \begin{cases} \exp\{-t|k|^{\alpha}[1 - i\beta\, \mathrm{tg}(\alpha\pi/2)\, \mathrm{sign}\, k]\}, & \alpha \neq 1, \quad |\beta| \leq 1 \\ \exp\{-t|k|\}, & \alpha = 1, \quad \beta = 0 \end{cases} \tag{5}$$

and

$$\varphi_C(k, t; \alpha, \beta) = \exp\{-t|k|^\alpha e^{-i\beta\alpha\pi/2 \operatorname{sign} k}\}, \qquad |\beta| \le \min\{1, 2/\alpha - 1\} \tag{6}$$

When $\alpha = 2$ the forms coincide,

$$\varphi_A(k, t; 2, \beta) = \varphi_C(k, t; 2, 0) = \exp\{-t|k|^2\}$$

and describe the Wiener process (3). When $\alpha = 1$ and $\beta = 0$ the characteristic functions

$$\varphi_A(k, t; 1, 0) = \varphi_C(k, t; 1, 0) = \exp\{-t|k|\}$$

correspond to the Cauchy process,

$$p_{A,C}(x, t; 1, 0) = \frac{t}{\pi(t^2 + x^2)}$$

If $\alpha = 1/2$ and $\beta_A = \beta_C = 1$, we have the Smirnov process,

$$p_A(x, t; 1/2, 1) = \frac{t}{\sqrt{2\pi}} x^{-3/2} \exp\left\{-\frac{t^2}{2x}\right\}$$

$$p_C(x, t; 1/2, 1) = \frac{t}{2\sqrt{\pi}} x^{-3/2} \exp\left\{-\frac{t^2}{4x}\right\}$$

As is readily seen from (5) and (6), the characteristic functions satisfy the evolution equations

$$\partial\varphi_A(k, t; \alpha, \beta)/\partial t = -|k|^\alpha[1 - i\beta \operatorname{tg}(\alpha\pi/2) \operatorname{sign} k]\varphi_A(k, t; \alpha, \beta), \qquad \alpha \neq 1 \tag{7}$$

$$\partial\varphi_A(k, t; 1, 0)/\partial t = -|k|\varphi_A(k, t; 1, 0)$$

and

$$\partial\varphi_C(k, t; \alpha, \beta)/\partial t = -|k|^\alpha e^{-i\beta\alpha\pi/2 \operatorname{sign} k}\varphi_C(k, t; \alpha, \beta) \tag{8}$$

with the initial condition

$$\varphi_A(k, 0; \alpha, \beta) = \varphi_C(k, 0; \alpha, \beta) = 1$$

3. EVOLUTION EQUATIONS FOR STABLE PROCESSES WITH $\alpha < 1$

Equation (7) can be rewritten in the form

$$\frac{1 + i\beta \operatorname{tg}(\alpha\pi/2) \operatorname{sign} k}{|k|^\alpha[1 + \beta^2 \operatorname{tg}^2(\alpha\pi/2)]}\frac{\partial\varphi_A}{\partial t} = -\varphi_A$$

Assuming

$$\delta^2 = [1 + \beta^2 \operatorname{tg}^2(\alpha\pi/2)] \cos(\alpha\pi/2)$$

and writing $\hat{F}p_A$ for φ_A [see (A.14)], we get

$$\frac{\cos(\alpha\pi/2) + i\beta \sin(\alpha\pi/2) \operatorname{sign} k}{|k|^\alpha \delta^2} \hat{F} \frac{\partial p_A}{\partial t} = -\hat{F}p_A$$

Comparing the left-hand side of this equality with the Fourier transform of Feller's potential (A.17) and inverting the transform, we arrive at the eqiation

$$M^\alpha_{a,v} \frac{\partial p_A}{\partial t} = -p_A(x, t; \alpha, \beta)$$

or

$$\frac{\partial p_A}{\partial t} = -(M^\alpha_{u,v})^{-1} p_A(x, t; \alpha, \beta) \tag{9}$$

with

$$u = \frac{1 + \beta}{2\delta^2}$$

and

$$v = \frac{1 - \beta}{2\delta^2}$$

We use the symbols for fractional integrodifferentiation operator according to the book of Samko *et al.* (1993). For the sake of convenience, they are given in the Appendix.

According to (A.11), the evolution equation (9) can be written in the following explicit forms:

$$\frac{\partial p_A(x, t; \alpha, \beta)}{\partial t} = -\frac{\alpha}{A\Gamma(1 - \alpha)} \int_{-\infty}^{\infty} \frac{1 + \beta \operatorname{sign}(x - \xi)}{|x - \xi|^{1+\alpha}} \times [p_A(x, t; \alpha, \beta) - p_A(\xi, t; \alpha, \beta)]\, d\xi \tag{10}$$

and

$$\frac{\partial p_A(x, t; \alpha, \beta)}{\partial t} = -\frac{\alpha}{A\Gamma(1 - \alpha)} \times \int_0^{\infty} [2p_A(x, t; \alpha, \beta) - (1 + \beta)p_A(x - \xi, t; \alpha, \beta) - (1 - \beta)p_A(x + \xi, t; \alpha, \beta)]\xi^{-1-\alpha}\, d\xi \tag{11}$$

where

$$A = [1 + \beta^2 \operatorname{tg}(\alpha\pi/2)]^{-1}$$

In the case of a symmetrical process ($\beta = 0$) the operator on the right-hand side of (11) coincides with the Riesz derivative (A.9),

$$\frac{\partial p_A(x, t; \alpha, 0)}{\partial t} = -D^{\alpha} p_A(x, t; \alpha, 0)$$

When $\beta = 1$ we have the one-sided stable process with the evolution equation

$$\partial p_A(x, t; \alpha, 1)/\partial t = -[\cos(\alpha\pi/2)]^{-1} \mathrm{D}_{+}^{\alpha} p_A(x, t, \alpha, 1)$$

where $\mathrm{D}_{+}^{\alpha} p_A$ is the fractional Marchoud derivative (A.6).

To transform equation (8) for the characteristic function to the corresponding equation for the density $p_C(x, t; \alpha, \beta)$ we rewrite it in the form

$$|k|^{-\alpha(1-\beta)} \hat{F} \frac{\partial p_C(x, t; \alpha, \beta)}{\partial t} = |k|^{\alpha\beta} e^{-i\alpha\beta\pi/2 \operatorname{sign} k} \hat{F} p_C(x, t; \alpha, \beta)$$

and use (A.16) and (A.19). As a result we have

$$I^{\alpha(1-\beta)} \frac{\partial p_C(x, t; \alpha, \beta)}{\partial t} = -D_{+}^{\alpha\beta} p_C(x, t; \alpha, \beta)$$

or

$$\frac{\partial p_C(x, t; \alpha, \beta)}{\partial t} = -D^{\alpha(1-\beta)} D_{+}^{\alpha\beta} p_C(x, t; \alpha, \beta) \tag{12}$$

In the symmetrical case ($\beta = 0$)

$$\frac{\partial p_C(x, t; \alpha, 0)}{\partial t} = -D^{\alpha} p_C(x, t, \alpha, 0)$$

In the extremely asymmetrical case ($\beta = 1$), $X(t; \alpha, 1) > 0$ if $\alpha < 1$, and (12) takes the form

$$\frac{\partial p_C(x, t; \alpha, 1)}{\partial t} = -D_{0+}^{\alpha} p_C(x, t; \alpha, 1) \tag{13}$$

where D_{0+}^{α} is given by (A.5). Performing the Laplace transformations, we obtain for

$$\tilde{p}_C(\lambda, t; \alpha, 1) = \int_0^{\infty} e^{-\lambda x} p_C(x, t; \alpha, 1)\, dx$$

the equation

$$\frac{\partial \tilde{p}_C(\lambda, t; \alpha, 1)}{\partial t} = -\lambda^{\alpha} \tilde{p}_C(\lambda, t; \alpha, 1)$$

Under the initial condition

$$\tilde{p}_C(\lambda, 0; \alpha, 1) = 1$$

we obtain

$$\tilde{p}_C(\lambda, t; \alpha, 1) = e^{-\lambda^{\alpha} t}$$

4. EVOLUTION EQUATIONS FOR STABLE PROCESSES WITH $\alpha > 1$

All the cases considered above concern the domain $\alpha < 1$. The transition to the domain $\alpha > 1$ can be performed with the help of the duality law. This refers to the relation connecting the strictly stable distribution with characteristic exponent α and the strictly stable distribution with $\alpha' = 1/\alpha$. In terms of stable random variables $Y(\alpha, \beta) \equiv Y_C(\alpha, \beta)$ it says that

$$\alpha \operatorname{Prob}\{Y(\alpha, \beta) > y\} = \operatorname{Prob}\{0 < Y(\alpha', \beta') < y^{-\alpha}\}$$

where $y > 0$, $\alpha \geq 1$, $\alpha' = 1/\alpha$, $\beta' = (1 + \beta)\alpha - 1$ (Zolotarev, 1986). According to (1), this relation can be rewritten in the form

$$\alpha \operatorname{Prob}\{X(t; \alpha, \beta) > x\} = \operatorname{Prob}\{0 < X(t'; \alpha', \beta') < (t'/x)^{\alpha} t\}$$

where t' is an arbitrary positive number. Considering it as a function of x and t and passing from probabilities to probability densities, we get the expression

$$\alpha \int_x^{\infty} p(x', t; \alpha, \beta)\, dx' = \int_0^{[t'(x,t)/x]^{\alpha} t} p(x', t'(x, t); \alpha', \beta')\, dx'$$

which after differentiating with respect to x takes the form

$$p(x, t; \alpha, \beta) = (t'/x)^{\alpha}[x^{-1} - (\partial t'/\partial x)/t']tp\,((t'/x)^{\alpha} t, t'; \alpha', \beta') \\ -\alpha^{-1} \int_0^{(t'/x)^{\alpha} t} \frac{\partial p(x', t', \alpha', \beta')}{\partial t'} \frac{\partial t'(x, t)}{\partial x}\, dx' \quad (14)$$

Equation (14) generalizes the known duality relation for stable distribution densities (Zolotarev, 1986),

$$g(x; \alpha, \beta) = x^{-1-\alpha} g(x^{-\alpha}; \alpha', \beta') \quad (15)$$

to stable processes. Setting $t' = t = 1$, we reduce (14) to (15). When t' does not depend on x, we have

$$p(x, t; \alpha, \beta) = (t'/x)^{\alpha}(t/x)p((t'/x)^{\alpha}t, t'; \alpha', \beta')$$

If

$$t'(x, t) = x$$

then

$$p(x, t; \alpha, \beta) = -\alpha^{-1} \int_0^t \frac{\partial p(x', x; \alpha', \beta')}{\partial x} dx' \tag{16}$$

Substitution of (2) into the right-hand side of (16) yields

$$\frac{\partial p(x', x; \alpha', \beta')}{\partial x} = -\alpha x^{-\alpha-1}\{g'\ (x'x^{-\alpha}; \alpha', \beta')x'x^{-\alpha} + g(x'x^{-\alpha}; \alpha', \beta')\}$$

and

$$\int_0^t \frac{\partial p(x', x; \alpha', \beta')}{\partial x} dx' = -\alpha x^{-1}\left\{\int_0^{tx^{-\alpha}} g'(z; \alpha', \beta')z\, dz + \int_0^{tx^{-\alpha}} g(z; \alpha', \beta')\, dz\right\}$$

where $g'(z; \alpha', \beta')$ is the derivative of $g(z; \alpha', \beta')$ with respect to z. After computing the first integral on the right-hand side by parts, we obtain

$$\int_0^t \frac{\partial p(x', x; \alpha', \beta')}{\partial x} dx' = -\alpha x^{-1-\alpha} tg(tx^{-\alpha}, \alpha', \beta') = -\alpha x^{-1} tp(t, x; \alpha', \beta') \tag{17}$$

Finally, the following duality relation results from (16) and (17):

$$xp(x, t; \alpha,\beta) = tp(t, x; 1/\alpha, (1 + \beta)\alpha - 1), \qquad \alpha \geq 1 \tag{18}$$

The use of the relation allows us to pass from evolution equations for $\alpha <$ 1 derived above to the equations for $\alpha > 1$. We will illustrate this by means of the following example.

Let us consider the Smirnov process, the density of which $p(x, t; 1/2, 1)$ satisfies (13) with $\alpha = 1/2$:

$$\frac{\partial p(x, t; 1/2,1)}{\partial t} = -D_{0+}^{1/2}\, p(x, t; 1/2, 1)$$

It obeys as well the equation

$$\frac{\partial^2 p(x, t; 1/2, 1)}{\partial t^2} = \frac{\partial p(x, t; 1/2, 1)}{\partial x} \tag{19}$$

The duality relation (18) in this case is of the form

$$xp(x, t; 2, 0) = tp(t, x; 1/2, 1)$$

Inserting it into (19) and exchanging the variables $t \leftrightarrow x$, we obtain

$$\partial[(x/t)p(x, t; 2, 0)]/\partial t = \partial^2[(x/t)p(x, t; 2, 0)]/\partial x^2$$

Simple calculations

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right) p(x, t; 2, 0) = \left(\frac{1}{t} - 4\frac{\partial}{\partial(x^2)}\right) p(x, t; 2, 0) = 0$$

lead to the Einstein equation, as was to be shown.

However, evolution equations for the symmetric stable processes $X(t; \alpha, 0) \equiv X(t; \alpha)$ with arbitrary $\alpha \in (0; 2]$ can be obtained immediately from the corresponding equations for characteristic functions and have the form

$$\partial p(x, t; \alpha)/\partial t = -D^\alpha\, p(x, t; \alpha), \qquad p(x, t; \alpha) \equiv p(x, t; \alpha, 0)$$

which stays valid in the n-dimensional case:

$$\partial p_n(x, t; \alpha)/\partial t = -(-\Delta_n)^{\alpha/2}\, p_n(x, t; \alpha), \qquad x \in \mathrm{R}^n$$

5. CONCLUDING REMARKS

This article has been stimulated by the works of Seshadri and West (1982), Allegrini *et al.* (1996), and West *et al.* (1997), where the evolution equation is given as follows [Eqs. (2.14), (37), and (53), respectively]:

$$\frac{\partial p(x, t)}{\partial t} = \text{const} \cdot \int_{-\infty}^{\infty} [1 + c\, \text{sign}(\xi - x)]\, \frac{p\,(\xi, t)}{|x - \xi|^{\alpha+1}}\, d\xi$$

This equation is evidently incorrect for all positive α because of the explicit divergence of the integral. Just the presence of the difference $p(x, t) - p(\xi, t)$ under the integral in the correct equation (10) provides for its convergence when $\alpha < 1$.

The correct fractional differential equations for the densities $g(x; \alpha, \beta)$ are obtained by Zolotarev (1986), but they differ in form from those given above because of the use of another kind of fractional derivative. Strictly speaking, they describe evolution of a complex function containing the stable density in its real part. This means that each of these equations is a system of two equations for two functions, of which only one is of interest to us. The linear nature of both equations allows one to write an equation for each of these functions, but only at the cost of complicating the equation. The approach developed here is free from this difficulty.

The equations derived in this article are applicable to the investigation of such anomalous diffusion processes as superdiffusion or enhanced diffusion which arise in some disordered media (Bouchaud and Georges, 1990; Isichenko, 1992).

APPENDIX. FRACTIONAL INTEGRODIFFERENTIAL OPERATORS

The Riemann–Liouville fractional integrals for $\alpha > 0$ are

$$(I_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{f(\xi)\, d\xi}{(x-\xi)^{1-\alpha}} \tag{A.1}$$

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(\xi)\, d\xi}{(x-\xi)^{1-\alpha}} \tag{A.2}$$

$$(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} \frac{f(\xi)\, d\xi}{(\xi-x)^{1-\alpha}} \tag{A.3}$$

The Riemann–Liouville fractional derivatives for $0 < \alpha < 1$ are

$$(D_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^{x} \frac{f(\xi)\, d\xi}{(x-\xi)^{\alpha}} \tag{A.4}$$

$$(D_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{0}^{x} \frac{f(\xi)\, d\xi}{(x-\xi)^{\alpha}} \tag{A.5}$$

$$(D_{-}^{\alpha} f)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\infty} \frac{f(\xi)\, d\xi}{(\xi-x)^{\alpha}}$$

The Marchaud fractional derivatives for $0 < \alpha < 1$ are

$$\begin{aligned}(\mathrm{D}_{+}^{\alpha} f)(x) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x) - f(x-\xi)}{\xi^{1+\alpha}}\, d\xi \\ &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f(x) - f(\xi)}{(x-\xi)^{1+\alpha}}\, d\xi \end{aligned} \tag{A.6}$$

$$(\mathrm{D}_{-}^{\alpha} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{0}^{\infty} \frac{f(x) - f(x+\xi)}{\xi^{1+\alpha}}\, d\xi \tag{A.7}$$

The Riesz potential for $\alpha > 0$, $\alpha \neq 1, 3, 5, \ldots,$ is

$$(I^{\alpha}f)(x) = \frac{1}{2\cos(\alpha\pi/2)}[(I_{+}^{\alpha}f)(x) + (I_{-}^{\alpha}f)(x)]$$
$$= \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}\frac{f(\xi)\,d\xi}{|x-\xi|^{1-\alpha}} \tag{A.8}$$

where I_{+}^{α} and I_{-}^{α} are given by (A.1) and (A.3), respectively.

The Riesz derivative for $0 < \alpha < 1$ is

$$D^{\alpha}f \equiv (I^{\alpha})^{-1}f$$
$$= \frac{\alpha}{2\Gamma(1-\alpha)\cos(\alpha\pi/2)}\int_{-\infty}^{\infty}\frac{f(x)-f(x-\xi)}{|\xi|^{1+\alpha}}\,d\xi$$
$$= \frac{\alpha}{2\Gamma(-\alpha)\cos(\alpha\pi/2)}\int_{0}^{\infty}\frac{2f(x)-f(x-\xi)-f(x+\xi)}{\xi^{1+\alpha}}\,d\xi$$
$$= [2\cos(\alpha\pi/2)]^{-1}(\mathrm{D}_{+}^{\alpha}f + \mathrm{D}_{-}^{\alpha}f) \tag{A.9}$$

where D_{+}^{α} and D_{-}^{α} are given by (A.6) and (A.7), respectively.

The Feller potential for $0 < \alpha < 1$ is

$$(M_{u,v}^{\alpha}f)(x) = u(I_{+}^{\alpha}f)(x) + v\,(I_{-}^{\alpha}f)(x)$$
$$= \int_{-\infty}^{\infty}\frac{u+v+(u-v)\,\mathrm{sign}(x-\xi)}{|x-\xi|^{1-\alpha}}f(\xi)\,d\xi \tag{A.10}$$

where $u^2 + v^2 \neq 0$. In particular,

$$M_{u,v}^{\alpha} = 2u\cos(\alpha\pi/2)I^{\alpha}$$

where I^{α} is given by (A.8).

The inverse Feller potential for $0 < \alpha < 1$ is

$$(M_{u,v}^{\alpha})^{-1}f = \frac{\alpha}{2A\Gamma(1-\alpha)}\int_{-\infty}^{\infty}\frac{u+v+(u-v)\,\mathrm{sign}(x-\xi)}{|x-\xi|^{1+\alpha}}$$
$$\times [f(x)-f(\xi)]\,d\xi$$
$$= \frac{\alpha}{2A\Gamma(1-\alpha)}\int_{0}^{\infty}[(u+v)\,f(x) - uf(x-\xi)$$
$$- vf(x+\xi)]\xi^{-1-\alpha}\,d\xi \tag{A.11}$$

where

$$A = [(u + v)\cos(\alpha\pi/2)]^2 + [(u - v)\sin(\alpha\pi/2)]^2$$

In particular,

$$(M^{\alpha}_{1,0})^{-1} = \mathrm{D}^{\alpha}_{+}$$

$$(M^{\alpha}_{0,1})^{-1} = \mathrm{D}^{\alpha}_{-}$$

$$(M^{\alpha}_{u,u})^{-1} f = [2u\cos(\alpha\pi/2)]^{-1} D^{\alpha} f$$

where D^{α} is given by (A.9).

The n-dimensional Riesz integrodifferentiation is given by

$$(-\Delta_n)^{-\alpha/2} f = \frac{1}{\gamma_n(\alpha)} \int_{R^n} \frac{f(\xi)\, d\xi}{|x - \xi|^{n-\alpha}} \tag{A.12}$$

where

$$\alpha > 0, \qquad \alpha \neq n, n + 2, n + 4, \ldots$$

$$\gamma_n(\alpha) = 2^{\alpha}\pi^{n/2}\, \Gamma(\alpha/2)/\Gamma((n - \alpha)/2)$$

and by

$$(-\Delta_n)^{\alpha/2} f = \frac{1}{d_{n,l}(\alpha)} \int_{R^n} \sum_{k=0}^{l} (-1)^k \binom{l}{k} f(x - k\xi)|\xi|^{-n-\alpha}\, d\xi \tag{A.13}$$

where

$$\alpha > 0, \qquad l = [\alpha] + 1$$

$$d_{n,l}(\alpha) = \frac{\pi^{1+n/2}}{2^{\alpha}\Gamma(1 + \alpha/2)\Gamma((n + \alpha)/2)\sin(\alpha\pi/2)} \sum_{k=0}^{l} (-1)^k \binom{l}{k} k^{\alpha}$$

In particular, if $n = 1$, then

$$\gamma_1(\alpha) = 2\Gamma(\alpha)\cos(\alpha\pi/2)$$

$$d_{1,1}(\alpha) = -2\Gamma(-\alpha)\cos(\alpha\pi/2), \qquad \alpha < 1$$

and the operators (A.12) and (A.13) coincide with (A.8) and (A.9), respectively.

The Fourier transforms

$$\hat{F}_n f \equiv \int_{R^n} e^{ik\cdot x} f(x)\, dx, \qquad \hat{F}_1 \equiv \hat{F} \tag{A.14}$$

are

$$\hat{F}(I^{\alpha}_{\pm} f) = |k|^{-\alpha} \exp\{\pm i\alpha(\pi/2)\,\mathrm{sign}\, k\}\hat{F}f, \qquad 0 < \alpha < 1 \tag{A.15}$$

$$\hat{F}(D^{\alpha}_{\pm} f) = |k|^{\alpha} \exp\{\mp i\alpha(\pi/2)\,\mathrm{sign}\, k\}\hat{F}f, \qquad \alpha \geq 0 \tag{A.16}$$

$$\hat{F}(M^{\alpha}_{u,v}f) = [(u + v)\cos(\alpha\pi/2) + i(u - v)\sin(\alpha\pi/2)\,\text{sign}\,k]\,|k|^{-\alpha}\hat{F}f, \qquad 0 < \alpha < 1 \tag{A.17}$$

$$\hat{F}_n((-\Delta_n)^{\alpha/2}f) = |k|^{\alpha}\hat{F}_n f \tag{A.18}$$

In particular,

$$\hat{F}_1((-\Delta_1)^{-\alpha/2}f) \equiv \hat{F}_1(I^{\alpha}f) = |k|^{-\alpha}\hat{F}_1 f \tag{A.19}$$

The Laplace transforms

$$\hat{L}f \equiv \int_0^{\infty} e^{-\lambda x} f(x)\,dx$$

are

$$\hat{L}(I^{\alpha}_{0+}f) = \lambda^{-\alpha}(\hat{L}f) \tag{A.20}$$

$$\hat{L}(D^{\alpha}_{0+}f) = \lambda^{\alpha}(\hat{L}f) \tag{A.21}$$

ACKNOWLEDGMENT

This work was supported by Russian Fund of Fundamental Research (Grant 98-01-03307).

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